16.1. Vector fields

Def A vector field is a function $\vec{F}$ which takes a point in $\mathbb{R}^2$ or $\mathbb{R}^3$ as an input and returns a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ as an output.

e.g. The gradient ∇f of a differentiable function f.

Force fields (gravitational, electric, magnetic, ⋯)

Velocity fields (of fluids)

Note We can study vector fields using their coordinate functions

- 2-dimension : $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
- 3-dimension : $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

Def Let $\vec{F}$ be a vector field.

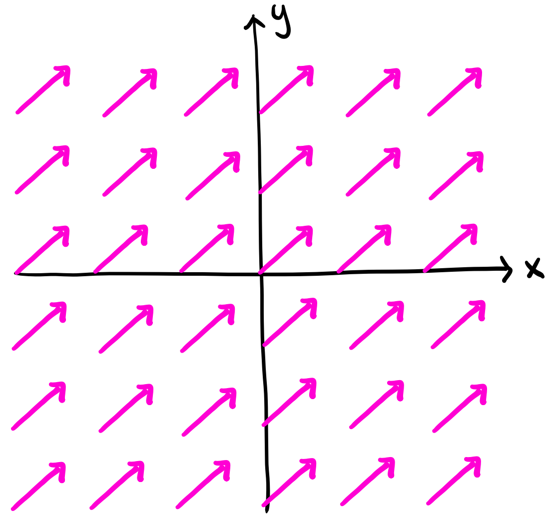
(1) $\vec{F}$ is continuous if its coordinate functions are all continuous.

(2) $\vec{F}$ is differentiable if its coordinate functions are all differentiable.

* In Math 215, we will only consider differentiable vector fields.

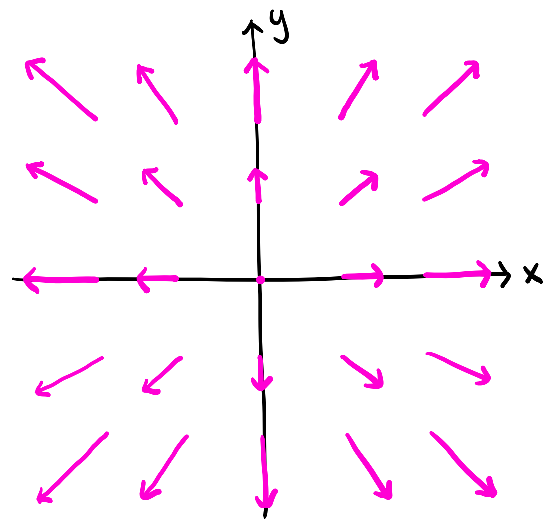
Note We can visualize a vector field by drawing arrows that represent output vectors.

e.g. (a) $\vec{F}(x,y) = (1,1)$



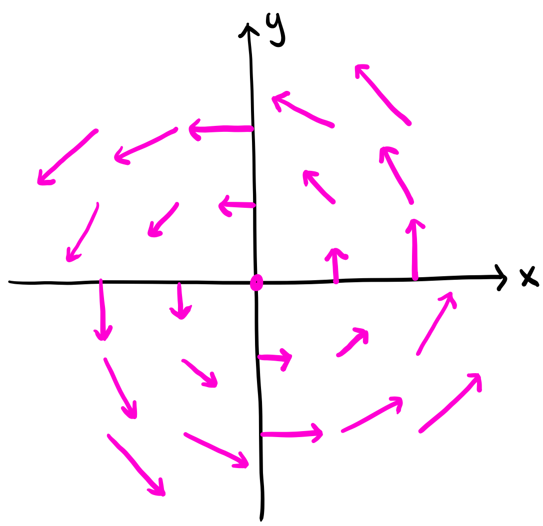
constant flows

(b) $\vec{F}(x,y) = (x,y)$.



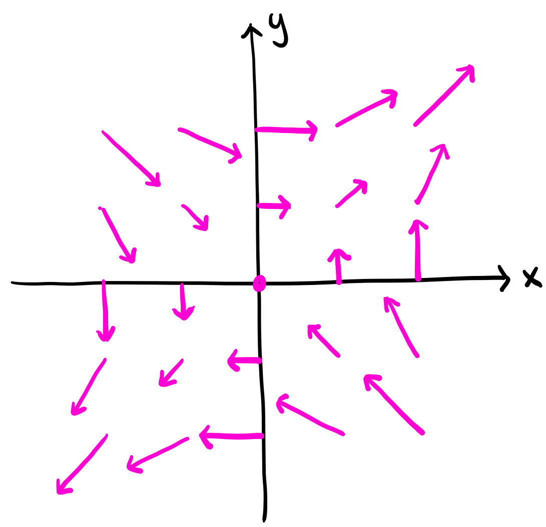
flows away from $(0,0)$

(c) $\vec{F}(x,y) = (-y,x)$



circular flows around $(0,0)$

(d) $\vec{F}(x,y) = (y,x)$.



flows towards the line $y=x$.

Ex Match each vector field with its plot.

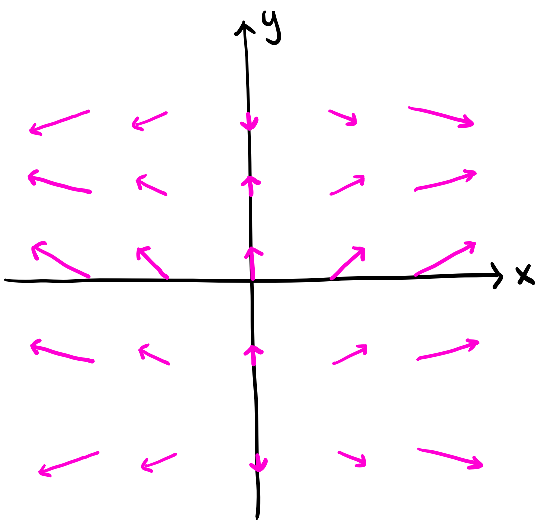
(a) $\vec{F}(x,y) = (y, x+y)$

(b) $\vec{G}(x,y) = (xy, y)$

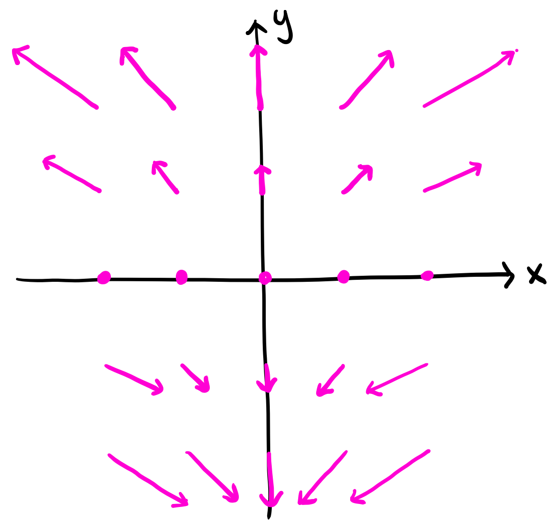
(c) $\vec{H}(x,y) = (x-1, x)$

(d) $\vec{I}(x,y) = (x, \cos y)$

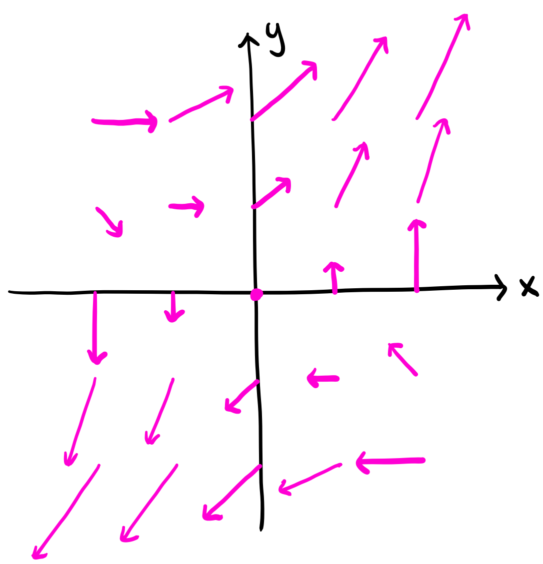
(I)



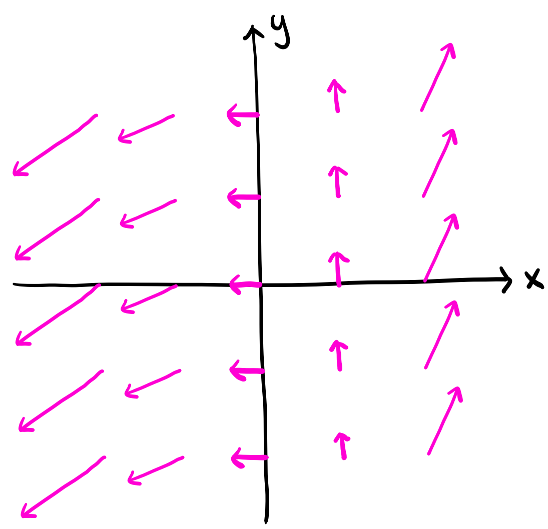
(II)



(III)



(IV)



Sol Idea : look for the points where the output vectors must be horizontal or vertical.

(a) $\vec{F}(x,y) = (y, x+y)$

$y=0$: $\vec{F}(x,y) = (0, x) \rightsquigarrow$ vertical

$x=-y$: $\vec{F}(x,y) = (y, 0) \rightsquigarrow$ horizontal

$\Rightarrow$ Match : (III)

(b) $\vec{G}(x,y) = (xy, y)$

$y=0$: $\vec{G}(x,y) = (0,0) \rightsquigarrow$ zero

$x=0$: $\vec{G}(x,y) = (0, y) \rightsquigarrow$ vertical.

$\Rightarrow$ Match : (II)

(c) $\vec{H}(x,y) = (x-1, x)$

$x=0$: $\vec{H}(x,y) = (-1, 0) \rightsquigarrow$ horizontal

$x=1$: $\vec{H}(x,y) = (0, 1) \rightsquigarrow$ vertical

$\Rightarrow$ Match : (IV)

(d) $\vec{I}(x,y) = (x, \cos y)$

$x=0$: $\vec{I}(x,y) = (0, \cos y) \rightsquigarrow$ vertical

$\Rightarrow$ Match : (I)

$\Rightarrow$ $\boxed{\text{(a) : (III), (b) : (II), (c) : (IV), (d) : (I)}}$